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The Sums of k-Fibonacci Numbers and k-Pell Numbers

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Abstract
In this paper, we present some identities for the sum of k-Fibonacci number and k-Pell numbers with $m+1$ consecutive terms of k- Pell numbers and the one for even, odd and their products and a close alternating sum of adjacent k- Pell numbers. Binet's formula is used to determine the characteristics of k- Pell numbers.
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1. Introduction

The two most well-known linear homogeneous recurrence relations of order two with constant coefficients are the Fibonacci and Pell sequences.

The well Known Fibonacci Sequence F_n is defined as
$F_0 = 0\,, F_1 = 1$,
$F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ (1.1)
In a similar way Pell sequence P_n is defined as
$P_0 = 0\,, P_1 = 1$,
$P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ (1.2)
Mainly, two methods have been used to generalize the second order recurrence sequence: first, by maintaining the initial conditions, and second, by maintaining the recurrence relation.

Let the k-Pell numbers as defined by Falcon [8] and consider how closely related they are to each other. The k-Pell sequence is defined by
$P_{k,0} = 0\,, P_{k,1} = k$,
$P_{\kappa,n+1} = 2kP_{k,n} + P_{k,n-1} \quad , where\ n \geq 1, \kappa \geq 1$ (1.3)

The first few terms of this Pell sequence are
$\{P_{k,n}\} = \{0, k, 2k^2, 4k^3 + k, 8k^4 + 4k^2, 16k^5 + 12k^3 + k, \dots\}$ (1.4)
The particular cases of the k-Pell Numbers are as follows.
If $k = 1$, the classical Pell sequence is obtained:
$\{0,1,2,5,12,29, \dots\}$ (1.5)
If $k = 2$, the Pell Lucas sequence is obtained:
$\{0,2,8,34,144,610, \dots\}$ (1.6)
The Binet's formulas for k −Pell numbers are given by

$$P_{k,n} = {R_1}^n + {R_2}^n$$

where R_1, R_2 are roots of characteristic roots of the recurrence equation

$$x^2 - kx - 1 = 0,$$

which are given by

$$R_1 = \frac{k + \sqrt{k^2+4}}{2},$$

$$R_2 = \frac{k - \sqrt{2k^2+4}}{2} \qquad (1.7)$$

$$\text{also } R_1 + R_2 = k,$$

$$R_1 R_2 = -1,$$

$$R_1 - R_2 = \sqrt{k^2+4}\,,\ {R_1}^2 - 1 = kR_1,\ {R_2}^2 - 1 = kR_2.$$

The Lucas triangle and its relationship to the k −Lucas numbers are presented by Falcon[7], along with the combinatorial formula for the k −Lucas numbers, the generating function, and the properties of the diagonals and rows of the triangle are defined. Falcon [8], study the properties of the k −Lucas numbers and will prove these properties will be related with the k −Fibonacci numbers. From a special sequences of squares of k −Fibonacci numbers, the k −Pell Sequences are obtained in a natural form. In addition, consider some of the intriguing characteristics of the k −Pell numbers and consider how closely they relate to the k −Fibonacci numbers. The k −Pell numbers have many characteristics with the k-Fibonacci numbers and are frequently found in many equations at the same time. Falcon [9], study the k −Pell numbers of arithmetic indexes of the form $an + r$ and present a formula for the sum of the square of the k −Fibonacci even numbers by mean of the k −Pell numbers.

2.1. Preliminary

Cerin [4], defines some sum of squares of odd and even terms of Lucas sequence. Rajesh and Leversha [20], define some properties of Fibonacci numbers in odd terms. Cerin [3], consider alternating sums of squares of odd and even terms of the Lucas sequence and alternating sums of their products. Cerin [5], improve some results on sums of squares of odd terms of the Fibonacci sequence by Rajesh and Leversha. Belbachir and Bencherif [1], recover and extend all result of Cerin [3], Cerin and Gianella [7]. Yazlik et al. [21], investigate some properties additive of the k −Fibonacci and the k −Pell sequences and obtain new identities on sums of powers these sequences and obtain the recurrence relations for powers of k −Fibonacci and k −Pell sequences. Also they will be given new formula for the powers of k −Fibonacci and the k −Pell sequences. Gnanam and Anitha [14], present some identities for the sums of squares of Fibonacci and Pell numbers with consecutive primes, using maximal prime gap $G(x) \sim \log^2 x,$ as indices. Panwar et al. [17], present the sum of consecutive members of k −Fibonacci numbers. Panwar and Gupta [18], define the sum of consecutive members of Fibonacci sequences and the same thing for even and for odd and their product of adjacent Fibonacci numbers. In this paper we present the sum of consecutive members of k −Pell numbers.

3. On the Sums of k −Pell Numbers

In this section, we prove the some identities for sums of a finite number of consecutive terms of k −Pell numbers.

Theorem 3.1. For $m \geq 0$ and $n \geq 0$ the following equality holds:

$$\sum_{i=0}^{m} P_{k,n+i} = \frac{1}{k}\left[P_{k,n+m} + P_{k,n+m+1} - P_{k,n} - P_{k,n-1}\right].$$

Proof. By Binet's formula we have

L.H.S. $\quad \sum_{i=0}^{m} P_{k,n+i} = \sum_{i=0}^{m} (R_1^{n+i} + R_2^{n+i})$

$$= \left[\frac{R_1^{n+m+1} - {R_1}^n}{R_1 - 1} + \frac{R_2^{n+m+1} - {R_2}^n}{R_2 - 1}\right]$$

$$= \left[\frac{(R_2-1)(R_1^{n+m+1}-{R_1}^n)+(R_1-1)(R_2^{n+m+1}-{R_2}^n)}{(R_1-1)(R_2-1)}\right]$$
$$= \left[\frac{R_1^{n+m}(R_1R_2)-R_1^nR_2-R_1^{n+m+1}+R_1^n+R_2^{n+m}(R_1R_2)-R_1R_2^n-R_2^{n+m+1}+R_2^n}{R_1R_2-R_1-R_2+1}\right]$$
$$= \left[\frac{-(R_1^{n+m}+R_2^{n+m})-(R_1^nR_2+R_1R_2^n)-(R_1^{n+m+1}+R_2^{n+m+1})+(R_1^n+R_2^n)}{-(R_1+R_2)}\right]$$
$$= \frac{1}{k}\left[-(R_1^{n+m}+R_2^{n+m})-(R_1^{n-1}+R_2^{n-1})+(R_1^{n+m+1}+R_2^{n+m+1})+(R_1^n+R_2^n)\right]$$
$$\sum_{i=0}^{m} P_{k,n+i} = \frac{1}{k}\left[P_{k,n+m}+P_{k,n+m+1}-P_{k,n}-P_{k,n-1}\right] = \text{R.H.S.}$$

Theorem 3.2. For $m \geq 0$ and $n \geq 0$ the following equality holds:
$$\sum_{i=0}^{m} P_{k,2n+2i} = \frac{1}{k}\left[P_{k,2n+2m+1}-P_{k,2n-1}\right]$$

Proof. By Binet's formula we have

L.H.S. $\sum_{i=0}^{m} P_{k,2n+2i} = \sum_{i=0}^{m}(R_1^{2n+2i}+R_2^{2n+2i})$
$$= \left[\frac{R_1^{2n+2m+2}-{R_1}^{2n}}{kR_1}+\frac{R_2^{2n+2m+2}-{R_2}^{2n}}{kR_2}\right]$$
$$= \frac{1}{k}\left[\frac{R_1R_2(R_1^{2n+2m+1}-R_2^{2n+2m+1})-(R_1^nR_2-R_1R_2^n)}{R_1R_2}\right]$$

$$\sum_{i=0}^{m} P_{k,2n+2i} = \frac{1}{k}\left[P_{k,2n+2m+1}-P_{k,2n-1}\right] = \text{R.H.S}$$

Theorem 3.3. For $m \geq 0$ and $n \geq 0$ the following equality holds:
$$\sum_{i=0}^{m}(-1)^iP_{k,n+i} = \frac{1}{k}\left[(-1)^i\left(P_{k,n+m+1}-P_{k,n+m}\right)-\left(P_{k,n-1}-P_{k,n}\right)\right]$$

Theorem 3.4. For $m \geq 0$ and $n \geq 0$ the following equality holds:
$$\sum_{i=0}^{m}(-1)^iP_{k,2n+2i} = \frac{1}{k^2+4}\left[P_{k,2n-2}-P_{k,2n+2m}+P_{k,2n}-P_{k,2n+2m+2}\right]$$
The sum of square of consecutive k −Pell number is treated in the following theorem.

Theorem 3.5. For $m \geq 0$ and $n \geq 0$ the following equality holds:
$$\sum_{i=0}^{m} P_{k,2n+2i}^2 = \frac{1}{k}\left[\left(P_{k,2n+2m+1}-P_{k,2n-1}\right)+k\{(-1)^{n+m+1}-(-1)^n\}\right]$$

Proof. By Binet's formula we have
$$\sum_{i=0}^{m} P_{k,n+i}^2 = \sum_{i=0}^{m}\ (R_1^{n+i}+R_2^{n+i})^2$$
$$= \sum_{i=0}^{m}\left[(R_1^{2n+2i}+R_2^{2n+2i}+2(R_1R_2)^{n+i}\right]$$
$$= \left[R_1^{2n}\frac{R_1^{2m+2}-1}{R_1^2-1}+R_2^{2n}\frac{R_2^{2m+2}-1}{R_2^2-1}+2(R_1R_2)^n\frac{(R_1R_2)^{m+1}-1}{R_1R_2-1}\right]$$
$$= \left[\frac{R_1^{2n+2m+2}-R_1^{2v}}{kR_1^2}+\frac{R_2^{2m+2m+2}-R_2^{2n}}{kR_2^2}+2\left\{\frac{(-1)^{n+m+1}-(-1)^n}{R_1R_2-1}\right\}\right]$$
$$= \frac{1}{k}\left[\frac{R_1R_2\left(R_1^{2n+2m+1}+R_2^{2m+2m+1}\right)+\left(R_1^{2v-1}+R_2^{2n-1}\right)-k\{(-1)^{n+m+1}-(-1)^n\}}{R_1R_2}\right]$$
$$\sum_{i=0}^{m} P_{k,2n+2i}^2 = \frac{1}{k}\left[\left(P_{k,2n+2m+1}-P_{k,2n-1}\right)+k\{(-1)^{n+m+1}-(-1)^n\}\right]$$

In terms of k −Fibonacci numbers, identities on Sums of Squares of k −Pell numbers are found. The Falcon and Plaza [11,12,13], defined The k −Fibonacci numbers rely only on one integer parameter k, in the following way for any positive real number k, the $k-$ Fibonacci sequence is defined by

$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, n \geq 1$ with $F_{k,0} = 0,\ F_{k,1} = 1$.
The Binet's formula for k −Fibonacci sequence is given by

$$F_{k,n} = \frac{R_1^n - R_2^n}{R_1 - R_2}$$

In the same way we obtain the following results.

Theorem 3.6 For $m \geq 0$ and $n \geq 0$ the following equality holds:

$$\sum_{i=0}^{m} (P_{k,i}^2 + P_{k,i+1}^2) = \left(\frac{k^2+4}{k}\right) F_{k,2m+2}.$$

Theorem 3.7. For $m \geq 0$ and $n \geq 0$ the following equality holds:

$$\sum_{i=0}^{m} (P_{k,i+1} + P_{k,i-1}) = \left(\frac{k^2+4}{k}\right) \left[F_{k,m+1} + F_{k,m-1}\right].$$

Theorem 3.8. For $m \geq 0$ and $n \geq 0$ the following equality holds:

$$\sum_{i=0}^{m} (P_{k,v+1}P_{k,i} + P_{k,v}P_{k,i-1}) = \left(\frac{k^2+4}{k}\right) \left[(F_{k,m+v+1} + F_{k,m+v}) - (F_{k,v-1} + F_{k,v})\right].$$

4. Conclusion

In this paper we have investigated sum of k- Pell numbers. We presented and derived numerous identities. We define the sum of $m+1$ consecutive terms of k- Pell numbers, as well as the sum of $m+1$ consecutive terms of even and odd k- Pell numbers, and their product and alternating sums of consecutive k- Pell numbers.

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